

To the Issue of Justification of the Absolutely Continuous Spectrum Eigenfunctions Asymptotics in the Problem of Three One-Dimensional Short-Range Quantum Particles with Repulsion.

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1 Introduction

In the work [10] (based on [11]-[12]) there were introduced for the first time asymptotic formulas at infinity in coordinates for absolutely continuous spectrum eigenfunctions uniform at angular variables in the case of a quantum system of three one-dimensional particles with short-range repulsive pair potentials. The mentioned asymptotic formulas were obtained non-strictly within the framework of a fairly subtle heuristic analysis. Nevertheless the significance of such formulas in the context of the absence of proofs of the existence and uniqueness solution theorem for the given scattering problem was so essential that in the work [9] quite a successful attempt of numerical modelling was made, based on the results of the work [10].

Continuous spectrum eigenfunctions contain asymptotic contributions of different kinds of arising. As was shown (though on the heuristic level) in [10], in the case of one-dimensional particles the contributions of different kinds of arising have in the sense of weak asymptotics the same type of decrease at infinity in configuration space. The difference is only in there being or not singularities at angular variable in the diverging circular wave coefficient. Therefore a strict mathematical analysis of the situation seems especially complicated.

Exactly, as was shown, for example, in [10], the wave generated by three-particles process (scattering on the center) decreases as $x^{-1/2}$ with a smooth amplitude. Contributions generated by two-particles scattering processes (multiple re-scattering included) keep decreasing as $x^{-1/2}$, but already with a singular at angular variable (in the sense of weak asymptotics) amplitude.

Our goal is to describe these contributions on a strict mathematical level. With the aim in mind we give priority to studying resolvent limited values on absolutely continuous spectrum.

Considering the subject of this problem we must mention the works [3] and [8], where the absence of the Schroedinger operator singular continuous spectrum for a broad class of pair-wise interactions potentials is asserted. It means, in particular, that the limit values of the resolvent

$$w\text{-}\lim_{\varepsilon \searrow 0} R(E \pm i\varepsilon), \quad E \in (0, \infty)$$

of the operator considered here exist as absolutely continuous functions of variable E .

However, the Moore technique does not allow to find these limit values and moreover, to distinguish from them a full system of absolutely continuous spectrum eigenfunctions.

Within the frameworks of our approach which in principle is close to the fundamental work [5], the above mentioned limit values are calculated explicitly.

It is necessary to stress that the limitation of considerations for the case of finite pair potentials does not lead to a problem simplification in principle, as the interaction potential of all three particles remains non-decreasing at infinity but allows to set aside a certain number of technical details.

2 Preliminaries

2.1 Jacobi Coordinates. Reduction.

In the initial setting we consider a non-relativistic Hamiltonian H

$$H\psi = -\Delta\psi + \frac{1}{2} \sum_{1 \leq i \neq j \leq 3} v(z_i - z_j)\psi, \quad z_i \in \mathbb{R}, \quad \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3, \quad \psi = \psi(\mathbf{z}) \in \mathbb{C}, \quad (2.1)$$

Δ is a Laplace operator in \mathbb{R}^3 , v is an even¹ function $\mathbb{R} \rightarrow [0, +\infty)$, defining two-particles interaction².

For a better illustration we will regard the function v as a finite (and integrable) one. In the case the essential self-adjointness of the operator H in the square integrable functions space is well known, see, for example, [5]. We will describe more precise requirements to v a bit later.

The extension of the results obtained here to the case of arbitrary fast decreasing at infinity pair potentials, as well as Coulomb potentials, is to be discussed in subsequent publications.

To separate the center of mass motion, we confine Hamiltonian on the surface Π , defined by the equation $\sum z_i = 0$. Allowing here a certain negligence, we will hereafter define by Δ a Laplace-Beltrami operator on the plane $\Pi \subset \mathbb{R}^3$. As this takes place, on Π we can use any pair (x_i, y_i) , $i = 1, 2, 3$, of the so called Jacobi coordinates, defined by the equations

$$x_i = \frac{1}{\sqrt{2}}(z_k - z_j), \quad y_i = \sqrt{\frac{3}{2}}z_i,$$

the indices (i, j, k) here form even permutations. In view of the orthonormality of Jacobi coordinates and invariance of the Laplace-Beltrami operator we have

$$\Delta = \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2},$$

and our operator takes the form

$$H = -\Delta + V, \quad V = \sum_{i=1}^3 v_i, \quad v_i = v(x_i). \quad (2.2)$$

Note that the support of potential $\sum v_i$ lies in the infinite cross domain, the compact part of which is shown on the picture 2.1.

Define the resolvent of the operator H

$$R(\lambda) = (H - \lambda I)^{-1}, \quad \lambda \notin [0, +\infty),$$

From now on I is an identity operator. The resolvent $(H_0 - \lambda I)^{-1}$ of the unperturbed operator $H_0 = -\Delta$ will be defined by $R_0(\lambda)$.

¹ the requirement of evenness is not essential but is dictated by physical setting of the problem.

²in the setting a system of three quantum equal mass particles is considered.



Figure 1: The compact part of the operator H potential support.

2.2 Ultimate Absorption Principle

We are interested in the scattering problem within the framework of a so called stationary approach. In this case studying of wave operators is replaced by studying of the limit values $R(E \pm i0)$ of the resolvent of the operator in question, when a spectral parameter λ approaches the real axis $\lambda \rightsquigarrow E \pm i0$, $E \in (0, +\infty)$). The proof of the existence of such limit values in a suitable topology is the very essence of the ultimate absorption principle. Since the existence of resolvent limit values is established, studying of wave operators themselves follows the known standard scheme, see, for example, [13],[7].

As we will see, absolutely continuous spectrum eigenfunctions $\psi^\pm(x, E)$ can be reconstructed by the formula

$$R(x, y|E \pm i0) \underset{y \rightarrow \infty}{\sim} \psi^\pm(x, E) \cdot \frac{e^{\pm i\sqrt{E}|y|}}{\sqrt{|y|}},$$

compare, for example, [15].

2.3 Equipped Hilbert Space Method

The point in question is exactly about one of the main methods of topology introduction, in which limit values of the resolvent $R(E \pm i0)$, are considered, see [14],[7].

In the case into the main Hilbert space \mathcal{H} a certain Banach space \mathcal{B} is continuously embedded, which in its turn allows to embed \mathcal{H} into the conjugate to \mathcal{B} space \mathcal{B}^* and next to prove that $R(E \pm i\varepsilon) : \mathcal{B} \rightarrow \mathcal{B}^*$ has a continuous extension at $\varepsilon \downarrow 0$.

Thus generalized eigenfunctions in this case are treated as elements of the space \mathcal{B}^* , while the main object of study becomes a scalar product $(R(E \pm i\varepsilon)\varphi, \varphi)$, $\varphi \in \mathcal{B}$, at $\varepsilon \downarrow 0$.

Hereafter, to be definite, we will limit ourselves to the case $\text{Im}\lambda \downarrow 0$.

2.4 Friedrichs – Faddeev Model

In the Friedrichs - Faddeev model, see [5],[13], within the framework of the ultimate absorption principle an unperturbed operator is treated as an operator of multiplication by an argument, while the perturbation represents an integral operator with a smooth kernel. For applications it is as a rule connected with a transition to a momentum representation. Studying of resolvent analytical singularities in a momentum representation can prove to be easier than studying of its coordinate asymptotics. On the whole, we adopt this point of view too.

One of the main features of Faddeev approach [5] to many-body scattering problems is a transition of operator resolvent to a so called t - operator $T(\lambda)$, connected to the resolvent $R(\lambda)$ by the identity

$$T(\lambda) = V - VR(\lambda)V,$$

where V is a perturbation of the operator H_0 . To study the operator $T(\lambda)$ is considered the equation

$$T(\lambda) = V - VR_0(\lambda)T(\lambda),$$

where the resolvent is reconstructed according to the equality

$$R(\lambda) = R_0(\lambda) - R_0(\lambda)T(\lambda)R_0(\lambda).$$

In Friedrichs - Faddeev model an unperturbed resolvent “on continuous spectrum” serves as an operator of multiplication by a function with a pole. As a sequence, the kernel of the operator $T(\lambda)$ is assumed to be Hölder continuous. The matter is that while describing limit values of the above mentioned scalar product $(R(E \pm i0)\varphi, \varphi)$, there naturally arises a singular Hilbert integral so Hölder functions, contrary to functions only continuous, form function algebra, on which a singular Hilbert integral is defined as a bounded operator.

It is exactly the circumstance that will define our choice as well of the supplementary space, in the topology of which the resolvent values limited in the weak sense will be considered. Such space in an momentum representation will be the space $H^{\mu,\theta}(\mathbb{R}^2)$ ($\frac{1}{2} < \theta < 1, \frac{1}{2} < \mu < 1$) of Hölder functions with a norm

$$\|f\|_{\mu,\theta} = \sup_{\xi,\eta} (1 + |\xi|^{1+\theta}) \left(|f(\xi)| + \frac{|f(\xi + \eta) - f(\xi)|}{|\eta|^\mu} \right). \quad (2.3)$$

The function space in coordinate representation, Fourier images of which belong to $H^{\mu,\theta}(\mathbb{R}^2)$, we will define by $\hat{H}^{\mu,\theta}$.

In the work [5] contributions accounting from particle perturbations into the complete resolution was carried out on the basis of the analysis of so called Faddeev equations.

The analysis of resolvent singularities in a momentum representation is conveniently carried out within the framework of a so called alternating Schwartz method, see [4],[1],[2]. Note that the known Faddeev equations, see [5], can be interpreted as well as an alternating method version.

2.5 Alternating Schwartz Method

Naturally, first of all there stands out a possibility of separation of variables for a partial Hamiltonian

$$H_i = -\Delta + v_i. \quad (2.4)$$

Thereby, the existence of limit values of the resolvent $R_i(\lambda) = (H_i - \lambda I)^{-1}$ is easily controlled, or rather see below section 4.1.

As a sequence, there rises a question of accounting of such contributions into the resolvent $R(\lambda)$ of the complete Hamiltonian H with a total potential $\sum v_i$.

The scheme of such accounting is known in literature under the name of an alternating Schwartz method, see [4],[2]. Recall the main principles of the abstract version of the scheme, as a number of its details plays an important part in understanding of our computations.

Denote by $\{G_i\}_{i=1}^n$ a certain number of linear operators in a complex vector space \mathcal{X} . Define the operator

$$G = \sum_{i=1}^n G_i.$$

Assuming that all operators $I - G_i$ are bijective, put

$$I - \Gamma_i = (I - G_i)^{-1}. \quad (2.5)$$

The operators Γ_i bear a name of reflection operators.

The principle of the alternating Schwartz method resolves itself into the following

1) If the operator matrix

$$\mathbf{L} = \begin{pmatrix} I & \Gamma_1 & \dots & \Gamma_1 \\ \Gamma_2 & I & \dots & \Gamma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_n & \Gamma_n & \dots & I \end{pmatrix} \quad (2.6)$$

is a bijection of the space \mathcal{X}^n , then the operator $I - G$ is a bijection of the space \mathcal{X} .

2) Denoting by $\text{diag}(\Gamma_1, \dots, \Gamma_n)$ a corresponding diagonal matrix. In conditions of bijectivity (2.6) we will define $\boldsymbol{\gamma}$ as a solution of the equation

$$\mathbf{L} \cdot \boldsymbol{\gamma} = \text{diag}(\Gamma_1, \dots, \Gamma_n) \quad (2.7)$$

and introduce linear operators γ_{ij} in \mathcal{X} as components of the matrix $\boldsymbol{\gamma}$. The latter can be also characterized by the identities

$$\gamma_{ij} = \Gamma_i \left(\delta_{ij} I - \sum_{k \neq i} \gamma_{kj} \right) = \left(\delta_{ij} I - \sum_{k \neq j} \gamma_{ik} \right) \Gamma_j. \quad (2.8)$$

Then the reflection operator Γ , defined by the equation

$$I - \Gamma = (I - G)^{-1}, \quad (2.9)$$

is given in terms of γ_{ij} by the formula

$$\Gamma = \sum_{1 \leq i, j \leq n} \gamma_{ij} . \quad (2.10)$$

- 3) Of special significance is the fact that the components γ_{ij} can be reconstructed by the operator Γ by the formulas

$$\gamma_{ij} = -\delta_{ij}G_j - G_i(I - \Gamma)G_j . \quad (2.11)$$

- 4) At $n = 2$ the operator \mathbf{L} will be a bijection \mathcal{X}^2 then and only then, when the operator $(I - \Gamma_1\Gamma_2)$ will be a bijection \mathcal{X} .

At that

$$I - \Gamma = (I - \Gamma_2)(I - \Gamma_1\Gamma_2)^{-1}(I - \Gamma_1) . \quad (2.12)$$

Detailed proof of all the facts can be found, for example, in [2].

Note one more property which was not described in [2]. To be exact, the assertion of the fourth section can be generalized also for the case of an arbitrary number of perturbations:

- 5) if the operator $I - G$ together with the operators $I - G_i$ is a bijection of the space \mathcal{X} , then the operator \mathbf{L} will be a bijection of the space \mathcal{X}^n (with all ensuing consequences).

Indeed, the components of the inverse to \mathbf{L} operator, to be denoted by ω_{ij} , must satisfy the equation

$$\omega_{ij} + \Gamma_i \sum_{k \neq i} \omega_{kj} = \delta_{ij}I ,$$

which can be re-written in the form of

$$\omega_{ij} = (I - G_i)\delta_{ij} + G_i \sum_k \omega_{kj} .$$

But the sum $\omega_j = \sum_k \omega_{kj}$ is uniquely defined by the equation

$$(I - G)\omega_j = I - G_j ,$$

from which

$$\omega_{ij} = (I - G_i)\delta_{ij} + G_i(I - \Gamma)(I - G_j) .$$

Note that if the operator matrix γ in an appropriate Banach space is bounded and its norm is less than 1, then

$$\Gamma = \sum \Gamma_i - \sum_{i \neq j} \Gamma_i \Gamma_j + \sum_{i \neq j \neq k} \Gamma_i \Gamma_j \Gamma_k - \dots$$

where the series of sums converges in norm. The formula precisely explains the method name.

To enclose our problem into the scheme, we will need to single out a free resolvent

$$R_0(\lambda) = (-\Delta - \lambda I)^{-1} . \quad (2.13)$$

At that

$$R(\lambda) = R_0(\lambda)(I + \sum v_i R_0(\lambda))^{-1} . \quad (2.14)$$

3 Justification Scheme Description

3.1 Analytical Reasons of Existence of the Limit

To be definite, within the framework of the ultimate absorption principle we will follow only the passage to the limit $\text{Im } \lambda \downarrow 0$.

As can be easily seen from (2.14), we are determined to define $G_j(\lambda) = -v_j R_0(\lambda)$. Assuming that

$$R_j(\lambda) = (-\Delta + v_j - \lambda I)^{-1} = R_0(\lambda)(I + v_j R_0(\lambda))^{-1} = R_0(\lambda)(I - G_j(\lambda))^{-1} = R_0(\lambda)(I - \Gamma_j(\lambda)), \quad (3.1)$$

on account of $\Gamma_j(\lambda) = -G_j(\lambda)(I - G_j(\lambda))^{-1}$, we find

$$\Gamma_j(\lambda) = v_j R_j(\lambda). \quad (3.2)$$

One should keep in mind that the Neumann series

$$I - \Gamma_j(\lambda) = I + G_j(\lambda) + G_j^2(\lambda) + \dots$$

are unsuitable for studying the limit $\text{Im } \lambda \downarrow 0$, as the powers of $G_j(\lambda)$ in a momentum representation accumulate a singularity in the limit at the spectral parameter λ is approaching the real axis. Indeed, the operator v_j in a momentum representation has a kernel of the form

$$v_j(\mathbf{q}, \mathbf{q}') = \hat{v}(k_j - k'_j) \delta(p_j - p'_j),$$

where $\mathbf{q} = (k_j, p_j)$ is a dual to $\mathbf{z}_j = (x_j, y_j)$ variable, \hat{v} is in our case an entire function and $\delta(\cdot)$ is a delta function of Dirac. It is clear that the kernel of the operator $G_j^2(\lambda)$ will already have singularities of the form

$$\pi i \frac{\hat{v}(k_j - \sqrt{\lambda - p_j^2}) \hat{v}(\sqrt{\lambda - p_j^2} - k'_j) \delta(p_j - p'_j)}{\sqrt{\lambda - p_j^2} (k_j'^2 + p_j'^2 - \lambda)}$$

(with a properly chosen branch of the square root). The situation with a root singularity getting worse with every new iteration. It is exactly where the main difficulty of the mathematical scattering theory for the case of non-decreasing potentials is concealed.

In addition, the operators $\Gamma_j(\lambda)$ including their continuous spectrum limit values, can be described explicitly, as they belong to the problem allowing a variables separation and reduction to a one-dimensional scattering problem.

It is useful to disclose the mechanism which provides the weak limit existence of a product $R_j(\lambda) = R_0(\lambda)(I - \Gamma_j(\lambda))$ at $\text{Im } \lambda \downarrow 0$. The free resolvent in a momentum representation on the Hölder function space $H^{\mu, \theta}(\mathbb{R}^2)$ has a weak limit considered. If the function $v_j(\mathbf{q}, \mathbf{q}')$ were a smooth one ³ (the case of a fast decreasing potential in a coordinate representation), the operator $\Gamma_j(\lambda)$ would have a strong limit and then the product $R_0(\lambda)(I - \Gamma_j(\lambda))$ (recall that

³Hölderness is sufficient

the order of multipliers for the given assertion is important) would have a corresponding weak limit.

However in our case the operator $\Gamma_j(\lambda)$ has no strong limit mentioned. As we will show below (section 4.1), the product $\Gamma_j(\lambda)\varphi$, where $\varphi \in H^{\mu,\theta}(\mathbb{R}^2)$ in a momentum representation, has a singularity of the form

$$\Gamma_j(\lambda)\varphi(\mathbf{q}) \sim \frac{f(\mathbf{q})}{\sqrt{p_j^2 - \lambda}}, \quad (3.3)$$

where f is a Hölder function right up to the real value of the parameter λ . The corresponding contribution into $(R_j(\lambda)\varphi, \varphi)$ takes the form

$$\int dp_j \int dk_j \frac{f(\mathbf{q})\varphi(\mathbf{q})}{(p_j^2 + k_j^2 - \lambda)\sqrt{p_j^2 - \lambda}} \sim \int \frac{g(p_j) dp_j}{p_j^2 - \lambda}, \quad (3.4)$$

where g is a certain Hölder function right up to the real value of the parameter λ (in the formula we permitted ourselves to follow only the integral singularity). The last integral, as is known, has a limit value at $\text{Im } \lambda \downarrow 0$.

It is clear that in the procedure described above the analytical character of the square root $\sqrt{p_j^2 - \lambda}$ is important.

In the work we will demonstrate that also in the case of a complete resolvent $R(\lambda)$ the ultimate absorption principle is based on a similar analytical procedure. And exactly within the framework of the alternating scheme one can avoid the singularity accumulation which is seen in the Neumann series.

3.2 Guiding Idea: Coordinate Asymptotic at Infinity of the Product $G_i G_j$ Kernel

We introduce a representation of the resolvent $R(\lambda)$ in the form of $R(\lambda) = R_0(\lambda)(I - \Gamma(\lambda))$, where $R_0(\lambda)$ is a free resolvent, while the operator Γ is defined by the equality

$$I - \Gamma(\lambda) = (I - G(\lambda))^{-1}, \quad G(\lambda) = \sum G_i(\lambda), \quad G_i(\lambda) = v_i R_0(\lambda).$$

The construction of the operator $\Gamma(\lambda)$ will be conducted according to the Schwartz scheme, see above.

First of all note that the representation of the operator $G_i G_j$ in the form of the Neumann series (when such a thing is possible), demonstrates a usefulness of studying the square of the operator $G_i G_j$ (the iteration method). We will apply the principle (for the reasons described above) not to the operator $I - G$ itself, but to the matrix operator \mathbf{L} , see (2.6).

In our case all the operators depend on the parameter λ . We will let ourselves omit this dependence, especially in the algebraic computations.

Denote $\mathbf{L} = I + \mathbf{\Gamma}$,

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \Gamma_1 & \Gamma_1 \\ \Gamma_2 & 0 & \Gamma_2 \\ \Gamma_3 & \Gamma_3 & 0 \end{pmatrix}.$$

Then

$$\mathbf{L}^{-1} = (\mathbf{I} + \mathbf{\Gamma})^{-1} = (\mathbf{I} - \mathbf{\Gamma}^2)^{-1}(\mathbf{I} - \mathbf{\Gamma}). \quad (3.5)$$

The components of operator matrix $\mathbf{\Gamma}^2$ are certain sums of operators of the form $\Gamma_i \Gamma_j$ at $i \neq j$.

One should expect that the key to the inversion of matrix $\mathbf{I} - \mathbf{\Gamma}^2$ is a study of coordinate asymptotics at infinity of the products $\Gamma_j \Gamma_k$ kernels. A bit later we will demonstrate that an asymptotic behaviour at infinity of the operators Γ_i kernels in the case considered bears the same character as for the operators G_i , and it in its turn means that the character of the asymptotics mentioned for products is easily discerned already when studying a coordinate asymptotics at infinity of the product $G_j G_k$ kernel.

Recall that the asymptotics of the kernel $R_0(\mathbf{z}_1, \mathbf{z}_2 | \lambda)$ at $\text{Im } \lambda > 0$ and $|\mathbf{z}_1 - \mathbf{z}_2| \rightarrow \infty$ is described by the relation

$$R_0(\mathbf{z}_1, \mathbf{z}_2 | \lambda + i0) \sim \frac{e^{i\pi/4}}{2\sqrt{2\pi}} \frac{e^{i\sqrt{\lambda}|\mathbf{z}_1 - \mathbf{z}_2|}}{\sqrt{|\mathbf{z}_1 - \mathbf{z}_2|}}.$$

Then finding an asymptotics at infinity of the product $G_j(\lambda)G_k(\lambda)$ kernel resolves itself to studying an asymptotics of an integral of the type

$$v(x_j) \int_{\mathbb{R}^2} \frac{v(x_k) e^{i\sqrt{\lambda}(|\mathbf{z}_j - \mathbf{z}_k| + |\mathbf{z}_k - \mathbf{z}'|)} d\mathbf{z}_k}{\sqrt{|\mathbf{z}_j - \mathbf{z}_k|} |\mathbf{z}_k - \mathbf{z}'|} \quad (3.6)$$

at $y_j \rightarrow \pm\infty$. Here $\mathbf{z}_j = (x_j, y_j)$ and $\mathbf{z}_k = (x_k, y_k)$ are the previously introduced Jacobi coordinates, while the variable \mathbf{z}' (keeping in mind the character of the alternating series) can be regarded as equal to a Jacobi coordinate different from \mathbf{z}_k .

If in the integral we restrict ourselves to integrating by a sufficiently large but finite band (i.e. by the domain $x_k \in \text{supp } v$, $|y_k| < T$, $T \gg 1$), then the asymptotics of the integral at $y_j \rightarrow \pm\infty$ is easily computed on account of stabilization of the modulus $|\mathbf{z}_j - \mathbf{z}_k| \sim |y_j| \pm \mathbf{l}_j \cdot \mathbf{z}_k$ (signs are correlated), where \mathbf{l}_j is a unitary vector in the axis y_j direction. As can be easily seen, it will lead to a separation of the rank two operator, the behaviour of which at $\text{Im } \lambda \downarrow 0$ is easily controlled. The operator left after the asymptotic separation will have a kernel fairly fast decreasing by \mathbf{z}_j at infinity, and it leads to its compactness in the space $\hat{H}^{\mu', \theta'}$ at sufficiently small $\mu' > 0$ and $\theta' > 0$.

Now we should stress that the integral (3.6) is to be taken only as a conditionally convergent by y_k and this comment is exceptionally essential for accounting a contribution into the given integral from the domain $|y_k| > T$.

For the evaluation of such a contribution it is natural to resort to integrating by parts by the variable y_k , which leads to an absolute integral convergence and essentially reduces the matter to the separation already considered. Indeed, if we introduce a smooth cut-off function χ_T equal to a unit on the interval $(-T, T)$ with a support on the interval $[-T-1, T+1]$, and put $\chi'_T = 1 - \chi_T$, then a formal integration by parts in the integral

$$v(x_j) \int \frac{\chi'_T(y_k) v(x_k) e^{i\sqrt{\lambda}(|\mathbf{z}_j - \mathbf{z}_k| + |\mathbf{z}_k - \mathbf{z}'|)} d\mathbf{z}_k}{\sqrt{|\mathbf{z}_j - \mathbf{z}_k|} |\mathbf{z}_k - \mathbf{z}'|} \quad (3.7)$$

by the Jacobi coordinate y_k will accumulate a power decrease either by the variable $|\mathbf{z}_j - \mathbf{z}_k|$, or by the variable $|\mathbf{z}_k - \mathbf{z}'|$.

In the first case the contribution is related to asymptotic correction terms and will be an additional term to the compact part of the product $G_j G_k$. In the second case we can consider (accomplishing several integrations by parts) the decrease power by integration variable fairly large, which allows to separate a main term of the decomposition from the multiplier $|\mathbf{z}_j - \mathbf{z}_k|^{-1/2}$ (i.e. $|\mathbf{z}_j|^{-1/2}$) at $y_j \rightarrow \infty$.

Though this integration by parts, at first sight, is interfered with by the existence of a stationary point (a geometrical optics point of reflection or transition):

$$\left(\frac{\mathbf{z}_j - \mathbf{z}_k}{|\mathbf{z}_j - \mathbf{z}_k|} - \frac{\mathbf{z}_k - \mathbf{z}'}{|\mathbf{z}_k - \mathbf{z}'|} \right) \cdot \mathbf{l}_k = 0,$$

the contribution from which must be studied separately. However, the existence of such a stationary point relates only to the case of a standard improper integral with which we do not deal in the given case. We will show below that within the framework of the alternating Schwartz method the interpretation of the given integral as an iterated one leads effectively to the stationary point case in a standard sense being not realized, or, to be more exact, does not prevent a separation of the asymptotics of the already described type.

To analyze the contribution from a stationary point, it is useful to consider the off-the-screen ray reflection geometrical optics from the standard point of view, see picture 3.2. The ray transition case is analogous one. Generally speaking, the alternating Schwartz method which we use brings us to a study of the kernels asymptotic behaviour for chains of operators products (in the given case of the type $G_{i_1} G_{i_2} \dots G_{i_k}$) with pair-wise differing indices of neighbouring operators. It is sufficient for us to consider the products of threes. The matter is that as was shown already in [11],[10], there are no more than three interactions between the screens. It means that in the kernel of the product $G_j G_k G_i$ by $j \neq k \neq i$, taking into account a stationary point in the pair $G_j G_k$, — define the point by \mathbf{z}_{ji} — the pair $G_k G_i$ can be considered as a non-having a stationary point, which will let us integrate by parts by the variable y_i in the corresponding kernel part of the operator $G_j G_k G_i$

$$v(x_j) \int_{\mathbb{R}^4} \frac{\chi(\mathbf{z}_k) v(x_k) v(x_i) e^{i\sqrt{\lambda}(|\mathbf{z}_j - \mathbf{z}_k| + |\mathbf{z}_k - \mathbf{z}_i| + |\mathbf{z}_i - \mathbf{z}'|)} d\mathbf{z}_k d\mathbf{z}_i}{\sqrt{|\mathbf{z}_j - \mathbf{z}_k| |\mathbf{z}_k - \mathbf{z}_i| |\mathbf{z}_i - \mathbf{z}'|}}, \quad (3.8)$$

where $\chi(\mathbf{z})$ is a smooth finite function with a support in a fixed radius circle and equal identically to a unit in the vicinity of the stationary point \mathbf{z}_{ji} .

The integration by parts leads either to an integral absolute convergence by \mathbf{z}_k , which as above brings the situation to the already examined one, i.e. to the possibility to separate an asymptotics by y_j of the type described above, or the integration by parts makes the integral absolutely convergent by \mathbf{z}_i . It is noteworthy that the latter again brings us to a possibility to separate the asymptotics mentioned above, as in the case the variable \mathbf{z}_i can be regarded as defined on a compact domain, but as results from the reflections geometry, the variable \mathbf{z}_k must also be considered as defined on a compact domain, and it allows to separate the stated asymptotic behaviour.

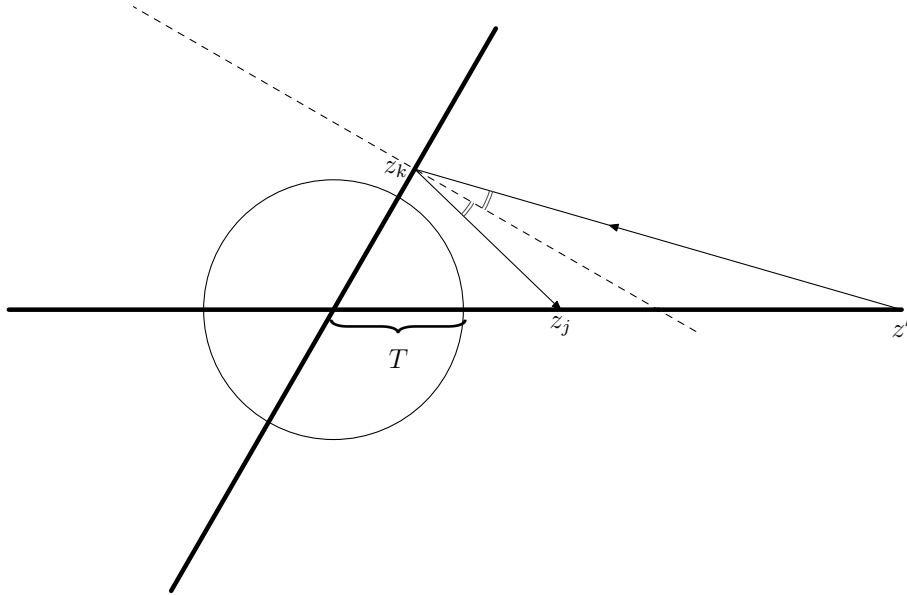


Figure 2: To the Question of Contribution from a Stationary Point

To show it strictly, note that considering the decrease by the variable z_i fairly strong (which we achieve by iterative integration by parts), we can change the order of integration, by making the integral by z_i an outer one. In the inner integral by z_k we can make use of the mean value theorem version which will put the variable z_k in the point in the vicinity of the reflection point z_{ji} , which in its turn means that we can start decomposing the function $|z_j - z_k|^{-1/2}$ by the Taylor formula by the degrees z_k , again separating the contribution into the asymptotics main term of the considered integral.

It is clear that the argument, in fact, leads us to a necessity to study the third degree of the matrix operator Γ , which will be done in the justifications section but now, a bit roughly, we restrict ourselves to the given considerations.

Thus, studying in the described key the kernel asymptotic of the operator $\Gamma_i(\lambda)\Gamma_j(\lambda)$ at $x \rightarrow \pm\infty$, we obtain a separation of the operator $\Gamma_j(\lambda)\Gamma_k(\lambda)$ into the sum $\Gamma_j(\lambda)\Gamma_k(\lambda) = A_{jk}(\lambda) + B_{jk}(\lambda)$, where $A_{jk}(\lambda)$ is a rank 2 operator, containing all the “bad” part of the product considered, i.e. the one which goes beyond the scope of the space $L_2(\mathbb{R}^2)$ at $\text{Im}\lambda \downarrow 0$, while the operator $B_{jk}(\lambda)$ is a compact operator in $\hat{H}^{\mu',\theta'}$, strongly continuous by $\lambda \text{Im}\lambda > 0$ and $0 < c_1 \leq \text{Re}\lambda \leq c_2 < \infty$, if $\mu' > 0$ and $\theta' > 0$ are small enough.

The limit operator $A_{jk}(E + i0)$ on the functions from $\hat{H}^{\mu',\theta'}$ acts into a two-dimensional functions space of the form

$$v(x_j)\varphi_j(x_j, 0)|y_j|^{-1/2}e^{i|y_j|\sqrt{E}}(C_1\chi_+(y_j) + C_2\chi_+(-y_j))$$

where $\chi_+(x)$ is defined as a smooth function equal identically to a unit at $x > T + 1$ and to a zero at $x < T$. The space of such functions has root analytical singularities on the axis in the impulse representation, especially singularities of the form $(p \pm \sqrt{E})^{-1/2}$. We will call the

described image of the operator $A_{jk}(\lambda)$ a functions space of the type A_j , while a vector space of such functions (by a fixed T) will denote by V_i .

Thus, the operator kernel $A_{jk}(\lambda)$ takes the form

$$\Phi_1(z_1|\lambda)\Psi_1(z_2|\lambda) + \Phi_2(z_1|\lambda)\Psi_2(z_2|\lambda),$$

where Φ_1, Φ_2 are functions of the type A_j . Note that the functions $\Psi_{1,2}(z_2|E+i0)$ at $|z_2| \rightarrow \infty$ behave as diverging waves

$$\Psi_{1,2}(z_2|E+i0) \sim \frac{e^{i\sqrt{E}|z_2|}}{\sqrt{|z_2|}}.$$

It should be stressed that it is the necessity of a separation of such an operator as A_{jk} that is an essential distinguishing feature of the given problem — the scattering problem of three one-dimensional particles — in comparison with a case of the scattering problem of three three-dimensional particles, considered in the work [5].

The sequence of the separation $\Gamma_i(\lambda)\Gamma_j(\lambda)$ is an effective (from the alternating scheme perspectives) representation

$$I - \Gamma^2(\lambda) = I - \mathbf{A}(\lambda) - \mathbf{B}(\lambda), \quad (3.9)$$

where the operator \mathbf{A} is of a finite rank, the components of which are composed of $A_{ij}(\lambda)$, while the components of the operator $\mathbf{B}(\lambda)$ are compact in $\hat{H}^{\mu',\theta'}$ with small enough positive μ' and θ' .

It is probably worth mentioning that the operators $A_{ij}(\lambda)$ or $B_{ij}(\lambda)$ are not components of operator matrices $\mathbf{A}(\lambda)$ or $\mathbf{B}(\lambda)$, correspondingly, in a sense of matrix analysis standard notations, but the components of the given operator matrices are composed of the operators $A_{ij}(\lambda)$ or $B_{ij}(\lambda)$ by the same rules, according to which the components Γ^2 are composed of the products $\Gamma_i\Gamma_j$.

3.3 The Structure of the Operator $I - \Gamma$

From here and from (3.5),(2.7) by way of a complimentary reduction based on the already described considerations, for the operator γ we obtain a representation

$$\gamma = (I - \Gamma)\text{diag}(\Gamma_1, \Gamma_2, \Gamma_3) + \mathbf{A}_2 + \mathbf{B}_2, \quad (3.10)$$

where \mathbf{A}_2 and \mathbf{B}_2 still possess the same properties as \mathbf{A} and \mathbf{B} .

As follows from (3.4), if φ_λ is a function of the type A_j , while $\psi \in \hat{H}^{\mu,\theta}$, then $(R_0(\lambda)\varphi_\lambda, \psi)$ has a limit at $\text{Im}\lambda \downarrow 0$. As a sequence, for the desired scalar operator $\Gamma(\lambda)$ constructions of the alternating Schwartz method lead to a representation

$$\Gamma(\lambda) = \sum \Gamma_i(\lambda) - \sum_{i \neq j} \Gamma_i\Gamma_j + N(\lambda), \quad N(\lambda) = A(\lambda) + B(\lambda), \quad (3.11)$$

where $A(\lambda)$ is a finite rank operator acting into the algebraic sum of class A_j spaces, while $B(\lambda)$ is a compact operator in $\hat{H}^{\mu,\theta}$, by $\mu, \theta > \frac{1}{2}$ (the change of these parameters range to be

discussed in the strict justifications section) and has a strong limit at $\text{Im } \lambda \downarrow 0$. At that the operator $N(\lambda)$ possesses the property that the product $R_0(\lambda)N(\lambda)$ has a weak limit in $\hat{H}^{\mu,\theta}$ at $\text{Im } \lambda \downarrow 0$ and $0 < c_1 \leq \text{Re } \lambda \leq c_2 < \infty$ for $\mu, \theta > \frac{1}{2}$. It means that $R(\lambda)$ has a weak limit in $\hat{H}^{\mu,\theta}$ at $\text{Im } \lambda \downarrow 0$ and $0 < c_1 \leq \text{Re } \lambda \leq c_2 < \infty$, and the limit is described by a weak equality

$$\begin{aligned} R(E + i0) &= R_0(E + i0)[I - \sum \Gamma_j(E + i0)] \\ &+ \sum_{i \neq j} R_0(E + i0)\Gamma_i(E + i0)\Gamma_j(E + i0) - \tilde{A}(E + i0) - \tilde{B}(E + i0) \\ &= R_0(E + i0) - \sum_{i \neq j} R_i(E + i0)\Gamma_j(E + i0) - \tilde{A}(E + i0) - \tilde{B}(E + i0), \end{aligned} \quad (3.12)$$

where $\tilde{A} = R_0 A$ is a finite rank operator, while $\tilde{B} = R_0 B$ is “compact”.

From the physical point of view the operator \tilde{A} describes a wave after at least three interactions with screens, while \tilde{B} — a wave reflected from the center.

In the following section we will turn the ideas into strict statements.

4 Justifications

4.1 Asymptotics of the Kernel Γ_j

From considerations of symmetry it is sufficient to describe a coordinate asymptotic of the kernel $\Gamma_1(\mathbf{z}, \mathbf{z}' | E + i0)$, when \mathbf{z} lies in the first band and \mathbf{z}' in the second. It is easiest to regard the kernel of a "two-particles" resolvent $R_1(\mathbf{z}, \mathbf{z}' | \lambda)$ as a convolution of

$$R_1(\mathbf{z}, \mathbf{z}' | \lambda) = \frac{1}{2\pi i} \int_C d\xi r(x, x' | \xi) r_0(y, y' | \lambda - \xi)$$

where r is a resolvent of one-dimensional problem with a potential $v(x)$, r_0 is a corresponding free resolvent, and C is a contour of integrating around the positive semi-axis in a standard negative direction, separating the point λ from the positive semi-axis. To prove the formula, introduce spectral measures $E(s)$ and $E_0(s)$ of the Schroedinger operators of the corresponding one-dimensional problems, then with an application of the residues theory

$$\frac{1}{2\pi i} \int_C d\xi \iint_{\mathbb{R}^2} \frac{dE(s)dE_0(t)}{(s - \lambda + \xi)(t - \xi)} = \iint_{\mathbb{R}^2} \frac{dE(s)dE_0(t)}{(s + t - \lambda)} = R_1(\lambda).$$

In connection with a position of the points $\mathbf{z} = (x, y)$ and $\mathbf{z}' = (x', y')$ we regard x' and $y - y'$ as large, while x is limited.

The free resolvent r_0 for the case $\text{Im } k > 0$ is given by the equality

$$r_0(y, y' | k^2) = \frac{i}{2k} e^{ik|y-y'|}.$$

Next, denote by φ_- a solution of the equation

$$-\varphi'' + v\varphi = k^2\varphi,$$

equal to $s(k)e^{ikx}$ at large positive x ($s(k)$ is transition coefficient and even function). Analogously, φ_+ is a solution of the same equation, equal to $s(k)e^{-ikx}$ at large negative x . The Jacobian of the solutions is equal to $W[\varphi_+, \varphi_-] = 2iks(k)$. The resolvent $r(x, x'|k^2)$ is equal to

$$\frac{\varphi_+(x)\varphi_-(x')}{W}, \quad x < x' \quad \text{and} \quad \frac{\varphi_-(x)\varphi_+(x')}{W}, \quad x > x'.$$

For the sake of definiteness consider the case when x' and $y - y'$ are large positive. Then

$$\begin{aligned} R_1(z, z'|\lambda) &= \frac{-1}{2\pi i} \int_C dk^2 \varphi_+(x, k) \frac{1}{4k\sqrt{\lambda - k^2}} e^{ikx' + i\sqrt{\lambda - k^2}(y - y')} \\ &= \frac{-1}{4\pi i} \int_{\mathbb{R}} \frac{\varphi_+(x, k) dk}{\sqrt{\lambda - k^2}} e^{ikx' + i\sqrt{\lambda - k^2}(y - y')}, \end{aligned}$$

where the branch of the square root $\sqrt{\lambda - k^2}$ is chosen with a positive imaginary component. The main contribution in the asymptotic of the present integral is brought by the saddle-point turning into a stationary one when the spectral parameter λ approaches the positive semi-axis. The point is described by the equation

$$x'\sqrt{\lambda - k^2} = k(y - y').$$

At that at the saddle-point

$$k = \frac{\sqrt{\lambda}x'}{|z - z'|} + O(|z - z'|^{-1}), \quad \sqrt{\lambda - k^2} = \frac{\sqrt{\lambda}(y - y')}{|z - z'|} + O(|z - z'|^{-1}),$$

and the phase function $\Phi(k) = kx' + \sqrt{\lambda - k^2}(y - y')$ is equal to $\sqrt{\lambda}(x'^2 + (y - y')^2) = \sqrt{\lambda}|z - z'| + O(1)$. The second derivative of the phase function is equal to

$$\Phi''(k) = \frac{\lambda(y - y')}{(\lambda - k^2)^{3/2}}$$

and at the saddle-point is equal to

$$\frac{1}{\sqrt{\lambda}} \frac{|z - z'|^3}{(y - y')^2} + O(1).$$

The saddle-point method (or the one of the stationary phase, if λ approaches the real axis) gives the asymptotic

$$R_1(z, z'|\lambda) = \varphi_+(x, k_0) \frac{e^{i\pi/4}}{2\sqrt{2\pi}\sqrt[4]{\lambda}} \frac{e^{i\sqrt{\lambda}|z - z'|}}{\sqrt{|z - z'|}} (1 + O(|z - z'|^{-1})) \quad (4.1)$$

(with our stated assumption concerning λ). Note that the asymptotics justifies the equation (3.3), as the asymptotics main order with an accuracy to a multiplier depending on the variable x coincides with the free resolvent asymptotics.

4.2 The Asymptotics of the Kernel $\Gamma_j \Gamma_k \Gamma_i$

In the subsection we will essentially repeat the arguments of the section 3.2.

Meaning the asymptotics (4.1), we will turn to the integral

$$I = \tilde{v}(x_j) \int_{\mathbb{R}^4} \frac{\tilde{v}(x_k) \tilde{v}(x_i) e^{i\sqrt{\lambda}(|z_j - z_k| + |z_k - z_i| + |z_i - z'|)} d\mathbf{z}_k d\mathbf{z}_i}{\sqrt{|z_j - z_k| |z_k - z_i| |z_i - z'|}}, \quad (4.2)$$

compare (3.8), where $\tilde{v} = v\varphi_+$ is a finite function. It is the integral that with an accuracy to a constant multiplier describes the asymptotics of the kernel $\Gamma_j \Gamma_k \Gamma_i$ at $y_j \rightarrow \infty$.

Thus, having fixed the large $T \gg 1$, we will make use of the separation of the unity $1 = \chi_T(y_k) + \chi'_T(y_k)$, $\chi'_T(x) = \chi_+(x) + \chi_+(-x)$, described in the section 3.2. In the integral

$$I_T = \tilde{v}(x_j) \int_{\mathbb{R}^4} \frac{\chi_T(y_k) \tilde{v}(x_k) \tilde{v}(x_i) e^{i\sqrt{\lambda}(|z_j - z_k| + |z_k - z_i| + |z_i - z'|)} d\mathbf{z}_k d\mathbf{z}_i}{\sqrt{|z_j - z_k| |z_k - z_i| |z_i - z'|}} \quad (4.3)$$

we easily separate the asymptotics at $y_j \rightarrow \infty$, as the function $|z_j - z_k|^{-1/2}$ can be decomposed in terms of powers of \mathbf{z}_k . At that (omitting the dependence on j, k, i)

$$I_T = A_T + B_T, \quad (4.4)$$

where A_T is a rank two operator with a kernel of the form

$$A_T(\mathbf{z}_j, \mathbf{z}') = v(x_j) \varphi_j(x_j, 0) |y_j|^{-1/2} e^{i|y_j|\sqrt{E}} (\chi_+(y_j) \psi_+(\mathbf{z}') + \chi_+(-y_j) \psi_-(\mathbf{z}')), \quad (4.5)$$

the functions ψ_{\pm} satisfy the estimate

$$|\psi_{\pm}(\mathbf{z})| \leq \frac{C}{\sqrt{1 + |\mathbf{z}|}}, \quad (4.6)$$

while the kernel of the operator B_T satisfies the estimate

$$|B_T(\mathbf{z}_j, \mathbf{z}')| \leq \frac{C|v(x_j)|}{(1 + |\mathbf{z}_j|)^{3/2} (1 + |\mathbf{z}'|)^{1/2}}. \quad (4.7)$$

A decomposition of the form (4.4) will be called the decomposition of the type AB .

In the integral

$$I'_T = \tilde{v}(x_j) \int_{\mathbb{R}^4} \frac{\chi'_T(y_k) \tilde{v}(x_k) \tilde{v}(x_i) e^{i\sqrt{\lambda}(|z_j - z_k| + |z_k - z_i| + |z_i - z'|)} d\mathbf{z}_k d\mathbf{z}_i}{\sqrt{|z_j - z_k| |z_k - z_i| |z_i - z'|}} \quad (4.8)$$

we will select a vicinity of the stationary point \mathbf{z}_{ji} by the variable \mathbf{z}_k , dividing the integral into a sum of integrals. Let I_p be the integral in which integrating by \mathbf{z}_k is carried out in this vicinity, and I_o be an additional integral.

In the integral I_o we can multiply integrate by parts by the variable y_k . Note that at $z = x + iy$ and $z_0 = x_0 + iy_0$

$$\begin{aligned}\frac{\partial}{\partial y}|z - z_0|^{-k/2} &= -\frac{k}{4}|z - z_0|^{-(k+4)/2}(y - y_0), \\ \frac{\partial^2}{\partial y^2}|z - z_0|^{-k/2} &= \frac{k^2}{16}|z - z_0|^{-(k+4)/2} - \frac{k(k+4)}{16}|z - z_0|^{-(k+8)/2}(x - x_0)^2.\end{aligned}$$

It means that with an accuracy to a term of the type B_T (with an estimate (4.7)) we will come to integrals of the form

$$\tilde{v}(x_j) \int_{\mathbb{R}^4} \frac{\chi'_T(y_k) \tilde{v}(x_k) \tilde{v}(x_i) e^{i\sqrt{\lambda}(|z_j - z_k| + |z_k - z_i| + |z_i - z'|)} dz_k dz_i}{\sqrt{|z_j - z_k|} |z_k - z_i|^N |z_i - z'|} \quad (4.9)$$

with fairly large positive N (and a bit overridden function $\tilde{v}(x_k)$), which allows to separate several terms in a decomposition of the function $|z_j - z_k|^{-1/2}$ by the powers of z_k . As a sequence, the integral I_o obtains a decomposition into a sum of the type AB , with the part B in the decomposition satisfying the estimate (4.7) with a constant C , which can be made arbitrarily small at $T \rightarrow \infty$.

In the integral I_p we will change the order of integration, making the integral by z_k an inner one. Then in the outer integral by the variable y_i we can integrate by parts, on account of the absence for it of the stationary point (recall that for the given geometry there cannot be two reflections). It means that we will come to a sum of two integrals of the form

$$\tilde{v}(x_j) \int \frac{\chi'_T(y_k) \tilde{v}(x_k) \tilde{v}(x_i) e^{i\sqrt{\lambda}(|z_j - z_k| + |z_k - z_i| + |z_i - z'|)} dz_k dz_i}{\sqrt{|z_j - z_k|} |z_k - z_i|^N |z_i - z'|^K}, \quad (4.10)$$

where $N + K$ is arbitrarily large and integration by z_k is extended onto the vicinity of the point z_{ji} . But as it follows in an elementary way from the geometry of reflections, $|z|_{ji} \leq C|z_i|$ with a certain constant C , not depending on the position of the point z_j . It will again let us separate several decomposition terms of the function $|z_j - z_k|^{-1/2}$ into the Taylor series which proves that the integral I_p is also decomposed in the sum of the type AB .

Thus we can state that the representation of the type AB is true:

$$\Gamma_i \Gamma_j \Gamma_k = A_{ijk} + B_{ijk}. \quad (4.11)$$

At that the procedure of an asymptotic separation of the product considered, described above, brings to light the following property useful for the furthest. The operator B_{ijk} can be represented in the form

$$B_{ijk} = \tilde{\Gamma}_i \tilde{\Gamma}_j \tilde{\Gamma}_k + E_{ijk}, \quad (4.12)$$

where $\tilde{\Gamma}_i$ has the same sense as the operator Γ_i but for a cut-off potential $v_i \chi_T$, i.e. for a finite one, and E_{ijk} satisfy the estimates (4.7), in which the constant C can be made arbitrarily small at $T \rightarrow \infty$.

4.3 Inversion of $I + \Gamma^3$

Meaning the equality

$$(I + \Gamma)^{-1} = (I - \Gamma + \Gamma^2)(I + \Gamma^3)^{-1}, \quad (4.13)$$

note that the procedure of the construction of the operator $I - \Gamma$ is based now on the inversion of the matrix operator $I + \Gamma^3$ with a control of the transfer to the limit at $\text{Im } \lambda \downarrow 0$ on the appropriate functions space.

From the equation (4.11) there follows a representation

$$I + \Gamma^3(\lambda) = I - \mathbf{A}(\lambda) - \mathbf{B}(\lambda), \quad (4.14)$$

where the operator \mathbf{A} is of the finite rank, the components of which are combined from $A_{ijk}(\lambda)$, while the components of the operator $\mathbf{B}(\lambda)$ are from B_{ijk} . The estimate (4.7), true for the components of the operator \mathbf{B} , demonstrate that the operator acts into the space of the functions $\hat{H}^{\mu, \theta}$ at $\mu < \frac{1}{2}$. To be able to regard the operator as a compact one, we must proceed from the space of the functions $\hat{H}^{\mu, \theta}$ with sufficiently small μ and θ . Unfortunately, the operators Γ_i in such spaces are not defined, to be more exact, we should require $\mu, \theta > \frac{1}{2}$. We will be helped by the fact that the operator acts into the space of the functions with a support in a band, and to search the inverse to $I + \Gamma^3$ is sufficient exactly for such functions.

Indeed, for $f \in L_q(\mathbb{R}^2)$ on account of the Hölder inequality

$$\left| \int_{\mathbb{R}^2} (1 + |z|)^{-1/2} f(z) dz \right| \leq \left(\int_{\mathbb{R}^2} (1 + |z|)^{-q'/2} dz \right)^{1/q'} \left(\int_{\mathbb{R}^2} |f(z)|^q dz \right)^{1/q} < \infty$$

at $q' > 4$ (q and q' are conjugate exponents), i.e. it is necessary that $q < \frac{4}{3}$. From the theorem B.1 (the second statement of the theorem) it follows that f from $\hat{H}^{\mu, \theta}$ lies in L_q for a certain $q < \frac{4}{3}$, if $\mu, \theta > \frac{1}{2}$. It is exactly the space that we fix.

The operator Γ_i at the limit value of the parameter λ maps the function from $\hat{H}^{\mu, \theta}$ in the function from $\hat{H}^{\nu, \theta}$ for $\nu > 0$, but at that the support of the latter lies in the band defined by the finite space of the potential v . In the Fourier images the action of the operator Γ_i on the function $\hat{\varphi} \in H^{\mu, \theta}$ will be defined by the equality

$$\begin{aligned} \Gamma_i \hat{\varphi}(k, p) &= \iint dk' dp' \delta(p - p') \tilde{v}(k - k') \frac{\hat{\varphi}(k', p')}{p'^2 + k'^2 - \lambda} \sim \frac{f(k, p)}{\sqrt{p^2 - \lambda}} \\ &= \frac{f(k, \sqrt{\lambda}) \eta(p)}{\sqrt{p^2 - \lambda}} + \frac{f(k, p) - f(k, \sqrt{\lambda}) \eta(p)}{\sqrt{p^2 - \lambda}}, \end{aligned} \quad (4.15)$$

where $f(p, k) = \pi i \tilde{v}(k - \sqrt{p^2 - \lambda}) \hat{\varphi}(\sqrt{p^2 - \lambda}, p)$, k, p are dual variables relative to x_i, y_i , the function \tilde{v} is (with an accuracy to a constant multiplier) a Fourier image of the product $v(x) \varphi_+(x, k_0)$, the function η is a smooth finite one, identically equal to a unit in the vicinity of the point $\sqrt{\lambda}$.

From the representation there follows that $\Gamma_i \hat{H}^{\mu, \theta}$ is a direct sum of spaces, one of which is two-dimensional and consists of the functions of A_i class, while the second one contains functions from $\hat{H}^{\nu, \theta}$, $\nu = \mu - \frac{1}{2} > 0$, with a support in a band. We will denote the latter by $\hat{H}_i^{\nu, \theta}$, and then $\Gamma_i \hat{H}^{\mu, \theta} \subset V_i + \hat{H}_i^{\nu, \theta}$. At the limit values of the parameter λ the sum is direct.

From the first part of theorem B.1 there follows that the function from $\hat{H}_i^{\mu, \theta}$ lies in L_r for $r < 2$ in the direction of the i -band. Thus the operator $\mathbf{\Gamma}^3$ will be regarded as an operator in the space $\mathbf{W}^{\nu, \theta} = \oplus_i (V_i + \hat{H}_i^{\nu, \theta})$.

To inverse the operator $I + \mathbf{\Gamma}^3$ we will use again the alternating Schwartz procedure.

First, the inversion of the operator $I - \mathbf{A}(\lambda)$ is controlled explicitly, as $\mathbf{A}(\lambda)$ is a finite rank operator. As we remember, the construction of the operator depended on the large parameter T . It is easy to verify that at sufficiently large T inverse operator $(I - \mathbf{A}(\lambda))^{-1} = I - \mathbf{\Gamma}_A(\lambda)$ exists and has a formal finite operator at $\text{Im } \lambda \downarrow 0$, acting into an algebraic sum of spaces of the type A_j , $j = 1, 2, 3$, i.e. into the space $\mathbf{V} = \oplus_i V_i$. It is proved in the appendix A.

Second, as was already noted, the operator $\mathbf{B}(\lambda)$ has the form $\mathbf{B}(\lambda) = \tilde{\mathbf{\Gamma}}^3(\lambda) + \mathbf{E}(\lambda)$, where the operator $\tilde{\mathbf{\Gamma}}(\lambda)$ has exactly the same sense as the operator $\mathbf{\Gamma}(\lambda)$, but for the finite potential $\tilde{V} = V \cdot \chi_T$, where χ_T is an arbitrary smooth finite function equal to a unit in the central circle of the radius T and having a support in the central circle of radius T , while the operator $\mathbf{E}(\lambda)$ is a small norm operator, if T is large enough.

The operator \mathbf{B} acts from $\mathbf{W}^{\nu, \theta}$ in $\oplus_i \hat{H}_i^{\sigma, \tau}$, where $\sigma < \frac{1}{2}$, $\tau > \frac{1}{2}$. At $\nu < \sigma$, $\theta < \tau$ it is compact.

Note now that the operator $\tilde{\mathbf{\Gamma}}^3$ does not have -1 as its eigenvalue right up to the limit values λ on the positive semi-axis. Moreover, a certain vicinity $|w + 1| < \delta$ of the point -1 will not contain the eigenvalues of the operator $\tilde{\mathbf{\Gamma}}^3$. Indeed, first note that on account of the equality

$$(H_0 - \lambda)(H_0 + \tilde{V} - \lambda - i0)^{-1} = I - \tilde{\mathbf{\Gamma}}(\lambda + i0) = (I - \tilde{G}(\lambda + i0))^{-1}$$

the compact operator \tilde{G} does not have a unit as its eigenvalue. Moreover, on account of

$$wI - \tilde{G}(\lambda) = wI - \tilde{V}R_0(\lambda) = w(I - w^{-1}\tilde{V}R_0(\lambda)),$$

the operator $\tilde{G}(\lambda + i0)$ does not have eigenvalues in a certain vicinity of the unit, as the only point of its spectrum is a zero. As a sequence, the operator $\tilde{\mathbf{\Gamma}}(\lambda + i0)$, analytically dependent on $\tilde{G}(\lambda + i0)$, does not have a unit as its eigenvalue with a whole vicinity and, indeed, the only point of its spectrum is zero.

On account of the alternating scheme the operator $\tilde{\mathbf{\Gamma}}$ has a zero point as the only point of spectrum. Indeed, if the point -1 were the point of spectrum for the operator, then for certain functions ϕ_1, ϕ_2, ϕ_3 from L_2 we would have

$$\tilde{\mathbf{\Gamma}}_i(\phi_j + \phi_k) = -\phi_i,$$

where the indices i, j, k form all even permutations. From here

$$\tilde{G}_i \sum \phi_j = \phi_i \quad \text{and} \quad \tilde{G}\phi = \phi,$$

where $\phi = \sum \phi_j$, which cannot be. If $w \neq 0$ were the eigenvalue of $\tilde{\Gamma}$, then the change $\tilde{\Gamma}_i$ for $-w^{-1}\tilde{\Gamma}_i$ would reduce it to the previous one (would bring us to the previous case).

But then the operator $\tilde{\Gamma}^3$ also has as its eigenvalue only the point zero, which exactly proves the statement made above.

As the operator norm \mathbf{E} can be made arbitrarily small at large T , from here there arises an invertibility $I - \mathbf{B}$. Thus, $I - \mathbf{B}(\lambda)$ has an inverse $I - \mathbf{\Gamma}_B(\lambda)$. Note that the inverse one has a limit in a strong operator sense at $\text{Im } \lambda \downarrow 0$.

Finally, the product $\mathbf{\Gamma}_A(\lambda)\mathbf{\Gamma}_B(\lambda)$ is once again the finite rank operator and the invertibility of the operator $I - \mathbf{\Gamma}_A(\lambda)\mathbf{\Gamma}_B(\lambda)$ does not require an involvement of new arguments in comparison with the invertibility of the operator $I - \mathbf{A}(\lambda)$, i.e. is a sequence of the considerations in appendix A.

According to the Schwartz method, we can conclude that at $\text{Im } \lambda > 0$ the operator $(I + \mathbf{\Gamma}^3)^{-1}$ has a representation

$$(I + \mathbf{\Gamma}^3)^{-1} = I - \mathbf{A}_1(\lambda) - \mathbf{B}_1(\lambda), \quad (4.16)$$

that is a decomposition of the type AB , i.e. the operators \mathbf{A}_1 and \mathbf{B}_1 have the same properties as the operators \mathbf{A} and \mathbf{B} , the matrix components $\mathbf{A}_1(\lambda)$ are the finite rank operators, acting into the algebraic sum of spaces of the type A_j , $j = 1, 2, 3$, , while the matrix components \mathbf{B}_1 are compact operators in $\mathbf{W}^{\nu, \theta}$, strongly continuous by λ at $\text{Im } \lambda \geq 0$ and $0 < c_1 \leq \text{Re } \lambda \leq c_2 < \infty$, if ν and θ are small enough.

4.4 The Structure of the Operators $I - \Gamma$ and $R(\lambda)$

Now we can proceed to a clarification of the provisions of the section 3.3. From the correlations (4.13), (4.14) we easily deduce that

$$\gamma = (I - \Gamma)\text{diag}(\Gamma_1, \Gamma_2, \Gamma_3) - \mathbf{N},$$

where \mathbf{N} allows a decomposition of the AB type. It fully justifies the representation (3.10) and, then, all the subsequent statements of the section 3.3.

It should be noted that the known effect of the wave function scattering amplitude discontinuity after re-propagations off the screens, see [11], is an effect of the second rank — it is completely controlled by the term $\sum_{i \neq j} R_i \Gamma_j$ in the formula (3.12).

A Inversion of the Finite-Dimensional Operator

We will consider here a procedure of a transformation of the operator

$$A = \begin{pmatrix} I - \varphi_1 \langle \psi_{12}, * \rangle - \varphi_1 \langle \psi_{13}, * \rangle & -\varphi_1 \langle \psi_{13}, * \rangle & -\varphi_1 \langle \psi_{12}, * \rangle \\ -\varphi_2 \langle \psi_{23}, * \rangle & I - \varphi_2 \langle \psi_{21}, * \rangle - \varphi_2 \langle \psi_{23}, * \rangle & -\varphi_2 \langle \psi_{21}, * \rangle \\ -\varphi_3 \langle \psi_{32}, * \rangle & -\varphi_3 \langle \psi_{31}, * \rangle & I - \varphi_3 \langle \psi_{31}, * \rangle - \varphi_3 \langle \psi_{32}, * \rangle \end{pmatrix}$$

By A_1 we will denote $I - A$ and instead of the operator A^{-1} we will search the operator $W = A^{-1} - I$, defined by the equation

$$A_1 W = W - A_1.$$

At that the operator A_1 has the form of

$$A_1 = \begin{pmatrix} \varphi_1 \langle \psi_{12}, * \rangle + \varphi_1 \langle \psi_{13}, * \rangle & \varphi_1 \langle \psi_{13}, * \rangle & \varphi_1 \langle \psi_{12}, * \rangle \\ \varphi_2 \langle \psi_{23}, * \rangle & \varphi_2 \langle \psi_{21}, * \rangle + \varphi_2 \langle \psi_{23}, * \rangle & \varphi_2 \langle \psi_{21}, * \rangle \\ \varphi_3 \langle \psi_{32}, * \rangle & + \varphi_3 \langle \psi_{31}, * \rangle & \varphi_3 \langle \psi_{31}, * \rangle + \varphi_3 \langle \psi_{32}, * \rangle \end{pmatrix}$$

It is natural to search the operator W in the form of:

$$W = \begin{pmatrix} \varphi_1 \alpha_{11} & \varphi_1 \alpha_{12} & \varphi_1 \alpha_{13} \\ \varphi_2 \alpha_{21} & \varphi_2 \alpha_{22} & \varphi_2 \alpha_{23} \\ \varphi_3 \alpha_{31} & \varphi_3 \alpha_{32} & \varphi_3 \alpha_{33} \end{pmatrix}$$

The functionals α_{ij} ($i, j = 1, 2, 3$) are to be determined. To be definite, we will write a system of equations for the first column elements.

$$\begin{aligned} \langle \psi_{12}, \varphi_1 \rangle \alpha_{11} + \langle \psi_{13}, \varphi_1 \rangle \alpha_{11} + \langle \psi_{13}, \varphi_2 \rangle \alpha_{21} + \langle \psi_{12}, \varphi_3 \rangle \alpha_{31} &= \alpha_{11} - \langle \psi_{12}, * \rangle - \langle \psi_{13}, * \rangle, \\ \langle \psi_{23}, \varphi_1 \rangle \alpha_{11} + \langle \psi_{21}, \varphi_2 \rangle \alpha_{21} + \langle \psi_{23}, \varphi_2 \rangle \alpha_{21} + \langle \psi_{21}, \varphi_3 \rangle \alpha_{31} &= \alpha_{21} - \langle \psi_{23}, * \rangle, \\ \langle \psi_{32}, \varphi_1 \rangle \alpha_{11} + \langle \psi_{31}, \varphi_2 \rangle \alpha_{21} + \langle \psi_{31}, \varphi_3 \rangle \alpha_{31} + \langle \psi_{32}, \varphi_3 \rangle \alpha_{31} &= \alpha_{31} - \langle \psi_{32}, * \rangle. \end{aligned}$$

Analogous are the systems of equations for functionals α_{j2} ($j = 1, 2, 3$) and α_{j3} ($j = 1, 2, 3$). For our appendices these systems are solvable on account of smallness of the coefficients — scalar products — at large T .

B L_p The Estimation of the Hölder Functions Fourier Images

Theorem B.1. *Let*

$$\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx = O(|h|^{\mu p}), \quad 1 < p < 2, \quad \frac{1}{p} - \frac{1}{2} < \mu < 1, \quad (\text{B.1})$$

then $\hat{f}(\xi)$ lies in L_q in any direction by $\frac{p}{p+\mu p-1} \leq q < 2$. If (B.1) is satisfied at

$$\min \left\{ 1, \frac{2n}{2+n} \right\} < p < 2, \quad n \left(\frac{1}{p} - \frac{1}{2} \right) < \mu < 1, \quad n \geq 2, \quad (\text{B.2})$$

then $\hat{f} \in L_q(\mathbb{R}^n)$ at $\frac{pn}{pn+\mu p-n} \leq q < 2$.

Indeed, the Fourier transform (by x) of the function $f(x+h)$ equals to $e^{i\xi \cdot h} \hat{f}(\xi)$ and from the boundness of the Fourier operator $L_p \rightarrow L_{p'}$ (p' is a conjugate exponent) we deduce

$$\int_{\mathbb{R}^n} \left| \sin \frac{\xi \cdot h}{2} \right|^{p'} |\hat{f}(\xi)|^{p'} d\xi = O(|h|^{\mu p'}).$$

Note that $|t| \leq C|\sin t|$ at $|t| < \frac{\pi}{2}$, from here

$$\int_{|\xi \cdot h| < 1} |\xi \cdot h|^{p'} |\hat{f}(\xi)|^{p'} d\xi = O(|h|^{\mu p'}) .$$

As a sequence

$$\int_0^{1/h_j} d\xi_j \xi_j^{p'} \int_{\mathbb{R}^{n-1}} d\tilde{\xi}_j |\hat{f}(\xi)|^{p'} = O(h_j^{(\mu-1)p'}) ,$$

where $\tilde{\xi}_j$ is supplementary to ξ_j component of the argument ξ and, for definiteness, $h_j > 0$. Symmetrizing the estimate, we will write

$$\int_0^{\xi_1} d\eta_1 \dots \int_0^{\xi_n} d\eta_n |\eta|^{p'} |\hat{f}(\eta)|^{p'} = O(|\xi|^{(1-\mu)p'})$$

(counting for definiteness all $\xi_j > 0$). Determine an auxiliary function

$$\varphi(\xi) = \int_1^{\xi_1} d\eta_1 \dots \int_1^{\xi_n} d\eta_n |\eta|^q |\hat{f}(\eta)|^q .$$

At that according to the Hölder inequality

$$\varphi(\xi) \leq \left(\int_1^{\xi_1} d\eta_1 \dots \int_1^{\xi_n} d\eta_n |\eta|^{p'} |\hat{f}(\eta)|^{p'} \right)^{q/p'} \prod \xi_j^{1-q/p'} \leq C |\xi|^{(1-\mu)q} \prod \xi_j^{1-q/p'} .$$

Then

$$\begin{aligned} \int_1^{\xi_1} d\eta_1 \dots \int_1^{\xi_n} d\eta_n |\hat{f}(\eta)|^{p'} &= \int_1^{\xi_1} d\eta_1 \dots \int_1^{\xi_n} d\eta_n \frac{1}{|\eta|^q} \frac{\partial^n \varphi(\eta)}{\partial \eta_1 \dots \partial \eta_n} \\ &= \frac{\varphi(\xi)}{|\xi|^q} + \sum (\pm) \int_1^{\xi_{j_1}} \dots \int_1^{\xi_{j_k}} d\eta_{j_k} \varphi(\zeta) \frac{\partial^k}{\partial \eta_{j_1} \dots \partial \eta_{j_k}} \frac{1}{|\zeta|^q} , \end{aligned} \quad (\text{B.3})$$

where a summation is carried out by all samples (j_1, \dots, j_k) , $1 \leq k \leq n$, at that $\zeta_i = \eta_i$ at i is equal to one of j_l in the considered sample and to $\zeta_i = \xi_i$ in the contrary case. Each term in the right-hand part (B.3) has the order

$$O(|\xi|^{-\mu q} \prod \xi_j^{1-q/p'}) .$$

At $1 - \frac{q}{p'} - \mu q \leq 0$, i.e. at $q \geq \frac{p}{p+\mu p-1}$, the integral in the left-hand part (B.3) by any one separated variable η_j remains bounded.

At $n(1 - \frac{q}{p'}) - \mu q \leq 0$, i.e. at $q \geq \frac{pn}{pn+\mu p-n}$, the whole even integral in the left in (B.3) remains bounded. The cases of other signs of variables ξ_j are analogous. The conditions on p and μ in (B.1) and (B.2) provide a non-emptiness of the variation intervals q . The theorem is proved.

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